SPINON BASIS FOR $(\widehat{\mathfrak{sl}_2})_k$ INTEGRABLE HIGHEST WEIGHT MODULES AND NEW CHARACTER FORMULAS

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ABSTRACT

In this note we review the spinon basis for the integrable highest weight modules of $\widehat{\mathfrak{sl}_2}$ at levels $k\geq 1$, and give the corresponding character formula. We show that our spinon basis is intimately related to the basis proposed by Foda et al. in the principal gradation of the algebra. This gives rise to new identities for the q-dimensions of the integrable modules.

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1. Introduction

In recent years, a number of problems in the area of condensed matter physics have been successfully analyzed with the help of Conformal Field Theory (CFT) techniques. Typical situations are those where one or more localized 'impurities' interact with, for example, a free electron gas or a gas of Quantum Hall edge excitations [1–3]. Some distance away from the impurity, such systems can be described by a fixed point of a renormalization group flow. Such fixed points are then described in terms of scale-invariant, or conformal, field theories.

For an analysis of this sort to be successful, it is essential that the CFT be formulated in a language that is appropriate for the physical situation at hand. In practice, this often means that one needs a description in terms of specific quasi-particles, which are singled out by the particular impurity coupling at the boundary.

While we know how to describe and solve a large class of interesting CFT's, the

description in terms of quasi-particles has only been developed in very special cases. In the conventional formulation of CFT, one exploits conformal invariance and the associated Ward identities. This language has the appeal of being universal to all CFT's, but it does not generalize away from the conformal point and, in general, it is not well adapted to a quasi-particle formulation. The same holds true for extended symmetries, such as affine (Kac-Moody) symmetries and $\mathcal{W}$-symmetries.

Recently, a number of CFT's have been reformulated in a quasi-particle language. As we already mentioned, these formulations do not refer to the chiral algebras that are usually used to describe rational CFT's. However, it has been found that, at least in special examples, other symmetry algebras come into play. These symmetries, which had gone unnoticed until recently, are fully compatible with a quasi-particle formulation. An example is the $SU(2)$ Wess-Zumino-Witten (WZW) model at level $k = 1$, for which a formulation in terms of quasi-particles (called 'spinons') was proposed in [4], see also [5–7]. The algebraic structure behind this description is the Yangian $Y(\mathfrak{sl}_2)$.

From the mathematical point of view, the reformulations of known CFT's are extremely interesting, since, in addition to suggesting new symmetry structures, they imply various identities that are obtained by equating quantities in different formulations. A standard example is the torus partition function, which can be decomposed into certain q-series called 'characters.' The reformulations thus lead to large numbers of character identities, which are usually highly non-trivial mathematically. Many of these q-identities have been explored by different means in the mathematical and theoretical physics literature [8–16]. For the example of the $SU(2)_1$ WZW model, the present authors used the spinon basis of [4] to derive alternative expressions for the Virasoro and affine characters in this theory [6].

In a recent paper [17], we generalized the results of [4,6] to the level $k > 1$ $SU(2)$ WZW models. We shall describe these result, which include an explicit spinon basis and new expressions for the characters, in section 2. In independent work [18], a spinon basis for the $\widehat{\mathfrak{sl}_2}$ Verma modules at generic level and highest weight has been obtained as a byproduct of the analysis of representations of an elliptic algebra called $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}_2})$. It was shown that for $k = 1$ this spinon basis reduces to a basis for the irreducible module. In section 3 below, we shall explain the relation between these results and our results in [17], and extend the results of [18] to the irreducible modules at level $k > 1$.

While the results presented below are mathematical in nature, they are directly relevant for at least two situations in physics. The first is the low energy behaviour of $SU(2)$ invariant spin chains of spin $s > \frac{1}{2}$ [19], and the second is the physics of the multichannel Kondo problem [1]. For both these situations the relevant CFT's are higher level SU(2) WZW models, and in both cases one may expect interesting

interpretations and applications of the spinon basis that we discuss here.

2. Spinon basis for the integrable highest weight modules of $(\widehat{\mathfrak{sl}_2})_k$

Let $(\widehat{\mathfrak{sl}_2})_k$ denote the (untwisted) affine Lie algebra, at level-k, associated to the simple finite dimensional Lie algebra $\mathfrak{sl}_2$ [20]. Let L_j denote the integrable (irreducible) highest weight modules of $(\widehat{\mathfrak{sl}_2})_k$ of spin $j = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2}$. Chiral Vertex Operators (CVO's) are, as usual, defined as the intertwiners

$$\Phi\begin{pmatrix} j_3 \\ j_2 \;\; j_1 \end{pmatrix} : \; L_{j_1} \otimes V_{j_3,z} \;\longrightarrow\; L_{j_2}\,, \tag{2.1}$$

where $V_{j,z} \cong V_j \otimes \mathbb{C}[z,z^{-1}]$ denotes the evaluation representation of the loop algebra $(\widehat{\mathfrak{sl}_2})_{k=0}$ associated to the finite dimensional irreducible $\mathfrak{sl}_2$ representation V_j of spin-j. Spinons are, by definition, the CVO's corresponding to $j_3 = \frac{1}{2}$. A CVO (2.1) exists (and is unique) iff j_2 occurs in the fusion rule $j_1 \times j_3$, *i.e.* $j_2 \in \{|j_1 - j_3|, \ldots, \min(j_1 + j_3, k - (j_1 + j_3))\}$. The basic relations for the CVO's are the so-called braiding and fusion relations, with braiding and fusion matrices satisfying the pentagon and hexagon identities (see *e.g.* [21]).

The CVO's $\Phi\begin{pmatrix} j_3 \\ j_2 \;\; j_1 \end{pmatrix}(z)$ have conformal dimension $\Delta(j_3)$ where

$$\Delta(j) \equiv \frac{j(j+1)}{k+2}\,, \tag{2.2}$$

and their mode expansion is given by

$$\Phi\begin{pmatrix} j_3 \\ j_2 \;\; j_1 \end{pmatrix}(z) = \sum_{n\in\mathbb{Z}} \Phi\begin{pmatrix} j_3 \\ j_2 \;\; j_1 \end{pmatrix}_{-n-(\Delta(j_2)-\Delta(j_1))} z^{n+(\Delta(j_2)-\Delta(j_1)-\Delta(j_3))}\,. \tag{2.3}$$

In general, due to the non-locality of the CVO's, the modes will not satisfy simple relations. If, however, the braiding and fusion matrices are one-dimensional ('abelian statistics'), the braiding and fusion relations can be combined into so-called generalized commutation relations and lead to parafermion-like algebras [22] (or Z-algebras [13]) known as 'generalized vertex algebras' [23]. This happens, for example, in the case of spinons at level $k = 1$, see [6,7].

Now, consider the Fock space of the spinons, *i.e.* the space spanned by the action of the spinon creation modes on the $\mathfrak{sl}_2$ singlet $|0\rangle$. Clearly, spinon monomials can be 'straightened' by means of the braiding and fusion relations. In [17] we argued that a set of (independent) basis vectors for the spinon Fock space $\mathcal{F}$, at level-k, is

provided by the states (at level $k=1$ this was proved in [6])

$$\begin{aligned}\phi^-\begin{pmatrix} & \frac{1}{2} & \\ j_{M+N} & & j_{M+N-1}\end{pmatrix}_{-\Delta_{M+N}-n_{M+N}} &\cdots\ \phi^-\begin{pmatrix} & \frac{1}{2} & \\ j_{M+1} & & j_M\end{pmatrix}_{-\Delta_{M+1}-n_{M+1}} \\ \times\, \phi^+\begin{pmatrix} & \frac{1}{2} & \\ j_M & & j_{M-1}\end{pmatrix}_{-\Delta_M-n_M} &\cdots\ \phi^+\begin{pmatrix} & \frac{1}{2} & \\ j_1 & & 0\end{pmatrix}_{-\Delta_1-n_1} |0\rangle\,,\end{aligned} \tag{2.4}$$

where the spins $\{j_1,\ldots,j_{M+N}\}$ run over the set of spins allowed by the fusion rules and we have put $\Delta_k=\Delta(j_k)-\Delta(j_{k-1})$. The modes $n_i\equiv n_{i,\min}+\tilde{n}_i$ satisfy $\tilde{n}_M\ \geq\ \tilde{n}_{M-1}\ \geq\ \ldots\ \geq\ \tilde{n}_1\ \geq\ 0$, $\tilde{n}_{M+N}\ \geq\ \tilde{n}_{M+N-1}\ \geq\ \ldots\ \geq\ \tilde{n}_{M+1}\ \geq\ 0$, where $n_{1,\min},\ldots,n_{M+N,\min}$ is a 'minimal allowed mode sequence' corresponding to the given Bratteli diagram, *i.e.* fusion channel, $(j_1,j_2,\ldots,j_{M+N})$ constructed as follows

$$\begin{aligned} n_{1,\min} &= 0\,, \\ n_{i+1,\min} &= \begin{cases} n_{i,\min}+1 & \text{if } j_{i+1}=j_{i-1}<j_i\,, \\ n_{i,\min} & \text{otherwise}\ . \end{cases}\end{aligned} \tag{2.5}$$

Moreover we argued that, as an $(\widehat{\mathfrak{sl}_2})_k$ module, $\mathcal{F}$ is in fact a direct sum of integrable highest weight modules. To be precise

$$\mathcal{F}\ \cong\ \bigoplus_{j=0}^{k/2}\ L_j\,, \tag{2.6}$$

where the state (2.4) belongs to L_j if and only if $j_{M+N}=j$. For example, the highest weight vectors $|j\rangle$ of L_j are given by

$$|j\rangle=\phi^+\begin{pmatrix} & \frac{1}{2} & \\ j & & j-\frac{1}{2}\end{pmatrix}_{-\Delta(j)+\Delta(j-\frac{1}{2})}\cdots\ \phi^+\begin{pmatrix} & \frac{1}{2} & \\ \frac{1}{2} & & 0\end{pmatrix}_{-\Delta(\frac{1}{2})}|0\rangle\,. \tag{2.7}$$

In [4] it was argued, by examining the infinite chain limit of the so-called Haldane-Shastry long-range spin chain model, that the integrable highest weight modules of $(\widehat{\mathfrak{sl}_2})_{k=1}$ carry a (fully reducible) representation of the Yangian $Y(\mathfrak{sl}_2)$. This was subsequently proven in [5,6] by utilizing the above spinon basis. In [17] it was argued that, in fact, the Yangian symmetry pertains at higher levels as well (see also [24,25]). We will not discuss this issue any further here.

In order to compute the resulting expressions for the characters, let us introduce, as usual, q-numbers and q-binomials by

$$(z;q)_N=\prod_{k=1}^{N}(1-zq^{k-1})\,,\qquad\qquad \begin{bmatrix} M \\ N\end{bmatrix}_q=\frac{(q;q)_M}{(q;q)_N(q;q)_{M-N}}\,, \tag{2.8}$$

and define for an arbitrary (symmetric) $k\times k$-matrix K and k-vector u, the q-series

$$\Phi_K^{m_1}(u;q) = \sum_{m_2,m_3,\ldots,m_k} q^{\frac{1}{4}m\cdot K\cdot m} \prod_{i\geq 2} \begin{bmatrix} \frac{1}{2}((2-K)\cdot m+u)_i \\ m_i \end{bmatrix}_q , \tag{2.9}$$

where the sum over $m_2,m_3,\ldots,m_k \in \mathbb{Z}_{\geq 0}$ obeys some restrictions that depend on the matrix K and vector u.

Given the spinon basis (2.4) the computation of the characters

$$\mathrm{ch}_{L_j}(z;q) = \mathrm{Tr}_{L_j}(q^{L_0} z^{J_0^3}) , \tag{2.10}$$

is straightforward. The sum over the spinon modes $\tilde{n}_1,\ldots,\tilde{n}_{M+N}$ contributes a factor

$$\mathcal{S}_{M,N}(z;q) = \frac{z^{\frac{1}{2}(M-N)}}{(q;q)_M (q;q)_N} , \tag{2.11}$$

while the sum over Bratteli diagrams of length $m_1 = M+N$, with the minimal mode sequence $n_{1,\mathrm{min}},\ldots,n_{M+N,\mathrm{min}}$, such that $j_{M+N}=j$, gives a contribution

$$q^{-\frac{j}{2}-\frac{1}{4}m_1^2}\ \Phi_{A_k}^{m_1}(u_j;q) , \tag{2.12}$$

where A_k is the Cartan matrix of the Lie algebra $A_k \cong \mathfrak{sl}_{k+1}$ and u_j is the unit vector $(u_j)_i = \delta_{i,2j+1}$. The summation in $\Phi_{A_k}^{m_1}(u_j;q)$ is over all odd positive integers for $m_{2j}, m_{2j-2}, m_{2j-4},\ldots$ and over the even positive integers for the remaining ones (we set $m_{k+1}\equiv 0$).

Thus, we have obtained the following expression for the characters

$$\mathrm{ch}_{L_j}(z;q) = q^{\Delta(j)-j/2} \sum_{M,N\geq 0} q^{-\frac{1}{4}(M+N)^2}\ \Phi_{A_k}^{M+N}(u_j;q)\ \mathcal{S}_{M,N}(z;q) . \tag{2.13}$$

For $k=1$ this reproduces the result of [9,5,6]. For $k>1$ this quasi-particle form of the character was first given, and checked up to high order in q, in [17]. Recently, the character formula was proved by examining the crystal basis of the integrable modules of $U_q(\widehat{\mathfrak{sl}_2})$ in the $q\to 0$ limit [24], as well as from the path description of the integrable $U_q(\widehat{\mathfrak{sl}_2})$ modules [25]. Evidently, the correctness of the characters (2.13) strongly supports the correctness of the basis (2.4).

For a discussion of the physical interpretation of the 'factorized' character (2.13), we refer to [17].

3. Spinon basis in the principal gradation

In this section we will discuss another, closely related, basis of the integrable highest weight modules of $(\widehat{\mathfrak{sl}_2})_k$ that was suggested in the work of Foda et al.

[18] on highest weight modules of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}_2})$. To this end, let us recall that the principal gradation of $(\widehat{\mathfrak{sl}_2})_k$ is defined by associating a degree $+1$ (the so-called 'q-dimension') to the Chevalley generators corresponding to the negative simple roots of $(\widehat{\mathfrak{sl}_2})_k$, *i.e.* to $J_{-1}^{(++)}$ and $J_0^{(--)}$ (for notation, see [17]), and degree -1 to the positive simple root generators. The q-dimension of a highest weight module V is the specialization of the (full) character, corresponding to the principal gradation (conventionally, the highest weight vector v of V is defined to have q-dimension zero). Unlike the homogeneous specialization (obtained by putting $z = 1$ the usual characters (2.10)) it is well defined for all highest weight modules (including Verma modules), because all spaces V_d, of q-dimension equal to d, are finite dimensional. Thus, we define

$$\dim_q V \;\equiv\; \sum_{d\geq 0} \dim(V_d)\, q^d\,. \tag{3.1}$$

The principal gradation of the spinon Fock space $\mathcal{F}$ can be implemented by making the following assignment of q-dimensions to our spinon operators

$$\begin{aligned}
\dim_q \phi^+\begin{pmatrix} \frac{1}{2} \\ j' \; j \end{pmatrix}_{-n-(\Delta(j')-\Delta(j))} &= \begin{cases} 2n & \text{if } j' = j+\frac{1}{2} \\ 2n-1 & \text{if } j' = j-\frac{1}{2} \end{cases} \\
\dim_q \phi^-\begin{pmatrix} \frac{1}{2} \\ j' \; j \end{pmatrix}_{-n-(\Delta(j')-\Delta(j))} &= \begin{cases} 2n+1 & \text{if } j' = j+\frac{1}{2} \\ 2n & \text{if } j' = j-\frac{1}{2} \end{cases}
\end{aligned} \tag{3.2}$$

It is now clear that upon defining a 'hybrid spinon operator' $\varphi\begin{pmatrix} \frac{1}{2} \\ j' \; j \end{pmatrix}_n$ by

$$\begin{aligned}
\varphi\begin{pmatrix} \frac{1}{2} \\ j+\frac{1}{2} \; j \end{pmatrix}_{-2n} &= \phi^+\begin{pmatrix} \frac{1}{2} \\ j+\frac{1}{2} \; j \end{pmatrix}_{-n-(\Delta(j+\frac{1}{2})-\Delta(j))} \\
\varphi\begin{pmatrix} \frac{1}{2} \\ j+\frac{1}{2} \; j \end{pmatrix}_{-(2n+1)} &= \phi^-\begin{pmatrix} \frac{1}{2} \\ j+\frac{1}{2} \; j \end{pmatrix}_{-n-(\Delta(j+\frac{1}{2})-\Delta(j))} \\
\varphi\begin{pmatrix} \frac{1}{2} \\ j-\frac{1}{2} \; j \end{pmatrix}_{-2n} &= \phi^-\begin{pmatrix} \frac{1}{2} \\ j-\frac{1}{2} \; j \end{pmatrix}_{-n-(\Delta(j-\frac{1}{2})-\Delta(j))} \\
\varphi\begin{pmatrix} \frac{1}{2} \\ j-\frac{1}{2} \; j \end{pmatrix}_{-(2n+1)} &= \phi^+\begin{pmatrix} \frac{1}{2} \\ j-\frac{1}{2} \; j \end{pmatrix}_{-n-1-(\Delta(j-\frac{1}{2})-\Delta(j))}
\end{aligned} \tag{3.3}$$

we have achieved that

$$\dim_q \varphi\begin{pmatrix} \frac{1}{2} \\ j' \; j \end{pmatrix}_{-n} \;=\; n\,. \tag{3.4}$$

We claim the following set of vectors provide an equivalent basis for the Fock space $\mathcal{F}$

$$\varphi\begin{pmatrix} \frac{1}{2} \\ j_N \; j_{N-1} \end{pmatrix}_{-n_N} \cdots \varphi\begin{pmatrix} \frac{1}{2} \\ j_2 \; j_1 \end{pmatrix}_{-n_2} \varphi\begin{pmatrix} \frac{1}{2} \\ j_1 \; 0 \end{pmatrix}_{-n_1} |0\rangle \tag{3.5}$$

where the spins $\{j_1, \ldots, j_N\}$ run over the set of spins allowed by the fusion rules. The modes $n_i \equiv n_{i,\mathrm{min}} + \tilde{n}_i$ satisfy $\tilde{n}_N \geq \tilde{n}_{N-1} \geq \ldots \geq \tilde{n}_1 \geq 0$, where $n_{1,\mathrm{min}}, \ldots, n_{N,\mathrm{min}}$ is a 'minimal allowed mode sequence' corresponding to the given fusion Bratteli diagram now constructed as follows

$$\begin{aligned} n_{1,\mathrm{min}} &= 0\,, \\ n_{i+1,\mathrm{min}} &= \begin{cases} n_{i,\mathrm{min}} + 1 & \text{if } j_{i+1} = j_{i-1}\,, \\ n_{i,\mathrm{min}} & \text{otherwise}\ . \end{cases} \end{aligned} \tag{3.6}$$

The state (3.5) belongs to L_j if and only if $j_N = j$.

Again, for level $k = 1$, we can rigorously prove that the states (3.5) provide a basis of $\mathcal{F}$ by means of the generalized commutation relations between the spinon modes.

For $k = 1$ we reproduce the basis of Foda et al. [18], discovered in the study of highest weight modules of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}_2})$. We believe that the spinon basis for Verma modules at generic level and highest weight, introduced in [18], reduces to (3.5) for integrable weights at level $k \in \mathbb{N}$. Thus, our basis (3.5) generalizes the $k = 1$ result of [18].

The computation of $\dim_q L_j$ using the basis of states (3.5) of q-dimension $d = \sum n_i$, is completely straightforward and analogous to the computation in section 2. The sum over Bratteli diagrams of length N with the minimal mode sequence $n_{1,\mathrm{min}}, \ldots, n_{N,\mathrm{min}}$, is now given by

$$q^{-\frac{1}{2}N(N+1)}\ \Phi^N_{A_k}(u_j; q^2)\,, \tag{3.7}$$

while the modes $\tilde{n}_i$ contribute a factor $(q;q)_N^{-1}$. We thus find

$$\dim_q L_j \ = \sum_{N\geq 0} \frac{q^{-\frac{1}{2}N(N+1)}}{(q;q)_N}\ \Phi^N_{A_k}(u_j; q^2)\,, \tag{3.8}$$

where the sum over N is over the even (odd) positive integers for j integer (half-integer).

Of course, (3.8) should be equal to the principal specialization of the character $\mathrm{ch}_{L_j}(z;q)$ (see (2.13)), *i.e.* we should have

$$\dim_q L_j \ = \ q^{-2\Delta(j)+j}\ \mathrm{ch}_{L_j}(q^{-1}; q^2)\,. \tag{3.9}$$

[The prefactor in (3.9) is chosen such that the highest weight vector of L_j has q-dimension zero, *i.e.* $\dim_q L_j = 1 + \mathcal{O}(q)$.] Indeed, equation (3.9) is straightforward to prove by making use of the following identity

$$\sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} q^n \ = \ \frac{(q^2;q^2)_N}{(q;q)_N} \ = \ (-q;q)_N\,. \tag{3.10}$$

By comparing (3.8) to the usual expression for the q-dimension of L_j (see *e.g.* [20], Proposition 10.10), *i.e.*

$$\dim_q L_j \;=\; \prod_{n\geq 1} \left(\frac{(1-q^{(k+2)n})(1-q^{(k+2)n+2j-k-1})(1-q^{(k+2)n-2j-1})}{(1-q^n)(1-q^{2n-1})} \right) , \quad (3.11)$$

we obtain a wealth of new q-identities (for $k=1$ this reproduces a specialization of the Cauchy identity, see *e.g.* [26] eq. (3.3.6)).

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